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Local Behaviour of the Deficient Discrete Cubic Spline Interpolator

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In this paper we have obtained a precise error estimate concerning deficient discrete cubic spline interpolant matching with the given function at the intermediate points between successive mesh points.

1. INTRODUCTION

Discrete splines have been introduced by Mangasarian and Schumaker [6] in connection with certain studies of minimization problems involving differences. Existence, uniqueness, and convergence properties of discrete cubic spline interpolant matching the given function at mesh points have been studied by Lyche [5] which have been further generalised by Dikshit and Powar [3] and Dikshit and Rana [4] for intermediate points of interpolation. It has been observed that deficient cubic splines are more applicable than usual splines as they require less continuity requirement at the mesh points. It has been observed by Boneva, Kendall, and Stefanov [2] that the local behavior of the derivative of a cubic spline interpolator is sometimes used to smooth a histogram which has been estimated in [8]. An asymptotically precise estimate of the difference between the discrete cubic spline interpolant and the function interpolated does not seem to be immediately available. In this connection Rana [7] has obtained a precise estimate concerning the discrete cubic spline interpolating the given function at the mesh points. In the present paper we obtain a similar precise estimate concerning the deficient discrete cubic spline interpolant matching the given function at two intermediate points between successive mesh points.

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2. Existence and Uniqueness

Let $P: 0 = x_0 < x_1 < \cdots < x_n = 1$ be a uniform partition of $[0, 1]$ such that $x_i - x_{i-1} = 1/n$ for $i = 1, 2, ..., n$. For a given $h > 0$ suppose a real continuous function $s(x, h)$ defined over $[0, 1]$ and its restriction to $[x_{i-1}, x_i]$ is a polynomial s_i of degree 3 or less for $i = 1, 2, ..., n$. Then $s(x, h)$ defines a deficient discrete cubic spline if

$$D_h^{\{j\}} s_i(x_i, h) = D_h^{\{j\}} s_{i+1}(x_i, h), \qquad j = 0, 1, \tag{2.1}$$

where for any function f and some $h > 0$, the central difference operator D_h is defined by

$$D_h^{\{0\}} f(x) = f(x), \qquad D_h^{\{1\}} f(x) = (f(x+h) - f(x-h))/2h,$$
$$D_h^{\{2\}} f(x) = (f(x+h) - 2f(x) + f(x-h))/h^2.$$

$S(3, P, h)$ denotes the class of all discrete deficient cubic splines which satisfies the periodicity condition

$$D_h^{\{j\}} s(x_0, h) = D_h^{\{j\}} s(x_n, h); \qquad j = 0, 1, 2. \tag{2.2}$$

Considering the following interpolatory conditions for a given function f,

$$s(\alpha_i) = f(\alpha_i); \qquad \alpha_i = x_{i-1} + 1/3n, \quad i = 1, ..., n-1, \tag{2.3}$$

$$s(\beta_i) = f(\beta_i); \qquad \beta_i = x_{i-1} + 2/3n, \quad i = 1, ..., n-1, \tag{2.4}$$

we shall prove the following.

Theorem 2.1. *Let f be 1-periodic. Then for any $h > 0$ there exists a unique 1-periodic deficient discrete cubic spline s in the class $S(3, P, h)$, which satisfies interpolatory conditions (2.3) and (2.4).*

Proof. Suppose in the interval $[x_{i-1}, x_i]$, for all i,

$$s(x, h) = AQ_{i-1}(x) - BQ_i(x) + CQ_i(x, \alpha) - DQ_{i-1}(x, \beta), \tag{2.5}$$

where

$$Q_i(x) = (x - x_{i-1})(x - \alpha_i)(x - \beta_i), \qquad Q_i(x, \alpha) = (x - x_{i-1})(x - \alpha_i)^2,$$
$$Q_{i-1}(x) = (x - x_i)(x - \alpha_i)(x - \beta_i), \qquad Q_{i-1}(x, \beta) = (x - x_i)(x - \beta_i)^2.$$

Now using (2.3)–(2.4) in (2.5), we determine constants C and D given by

$$C = 27n^3 f(\beta_i)/2; \qquad D = 27n^3 f(\alpha_i)/2$$

and setting $D_h^{\{1\}} s(x_i, h) = m_i(h) = m_i$, say for all i, we have

$$m_i = g(2)\,A - g(11)\,B + ((g(16)\,f(\beta_i) - g(1)\,f(\alpha_i))(27/2)\,n^3, \quad (2.6)$$

$$m_{i-1} = g(11)\,A - g(2)\,B + ((g(1)\,f(\beta_i) - g(16)\,f(\alpha_i))(27/2)\,n^3, \quad (2.7)$$

where for any real a, $g(a) = h^2 + a/9n^2$. Solving (2.6) and (2.7) for A, B and writing $g^*(a) = h^2 + g(a)$, $d(a) = g(a)/g^*(13)$, we substitute these values, along with the values of C and D in (2.5), to get

$$s(x, h) = n^2[m_i y_i + m_{i-1} y_i^1 + \tfrac{9}{2} n(f(\beta_i)\,\mathbf{Y}_i - f(\alpha_i)\,\mathbf{Y}_i^1)], \quad (2.8)$$

where $y_i = Q_i(x) - d(2)\{Q_i(x) + Q_{i-1}(x)\}$, $\mathbf{Y}_i = [d^*(7)\{Q_i(x) + Q_{i-1}(x)\} - 5Q_i(x) + 3Q_i(x, \alpha)]$ and $d^*(a) = g^*(a)/g^*(13)$. $\mathbf{Y}_i^1$ and y_i^1 are respectively obtained from the expressions of $\mathbf{Y}_i$ and y_i by interchanging $Q_i(x)$ with $Q_{i-1}(x)$ and $Q_i(x, \alpha)$ with $Q_{i-1}(x, \beta)$.

Now using the continuity condition (2.1) with $j = 0$, we have

$$-\frac{d(2)}{2} m_{i+1} + (1 - d(2))\,m_i - \frac{d(2)}{2} m_{i-1} = F_i, \qquad i = 1, 2, ..., n-1, \quad (2.9)$$

where $F_i = (9n/4)[(f(\alpha_{i+1}) - f(\beta_i)) + d^*(7)\{(f(\alpha_i) - f(\beta_i)) + (f(\alpha_{i+1}) - f(\beta_{i+1}))\}]$.

Existence and uniqueness of $s(x, h)$ depend on the existence of a unique solution of the set of Eq. (2.9) in m_i. It is easy to observe that in (2.9) absolute value of the coefficient of m_i dominates over the sum of the absolute values of the coefficients of m_{i+1} and m_{i-1}. Thus, the coefficient matrix of system of Eq. (2.9) is diagonally dominant and hence invertible.

3. Estimation of the Inverse of the Coefficient Matrix

Ahlberg, Nilson, and Walsh [1] have estimated precisely the inverse of the coefficient matrix appearing in the studies concerning continuous cubic spline interpolant matching the given function at the mesh points. We propose to obtain here a precise estimate for the inverse of the coefficient matrix of the system of Eq. (2.9) by using the approach studied in [1]. It may be mentioned that this method permits the immediate application to the spline in standard problems of numerical analysis (see [1]). Without loss of generality, we assume for the rest of this paper that the deficient discrete cubic spline $s(x, h)$ under consideration satisfies the condition $D_h^{\{1\}} s(x_0, h) = 0$. Now we write the system of Eq. (2.9) in the form

$$AM = F, \quad (2.10)$$

where M and F are column vectors $[m_1, ..., m_{n-1}]$ and $[F_1, ..., F_{n-1}]$, respectively, and the square matrix A is of order $n-1$.

Now in order to find the inverse of the coefficient matrix A, we define matrix $D_n(\alpha, \beta)$ of order n as

$$D_n(\alpha, \beta) = \begin{bmatrix} 2\beta & \alpha & 0 & 0 & 0 & 0 \\ \alpha & 2\beta & \alpha & 0 & 0 & 0 \\ 0 & \alpha & 2\beta & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \alpha & 2\beta & \alpha \\ 0 & 0 & 0 & 0 & \alpha & 2\beta \end{bmatrix},$$

where α and β are real numbers. It may be observed that the determinant $|D_n|$ satisfies the difference equation

$$|D_n(\alpha, \beta)| = 2\beta\, |D_{n-1}(\alpha, \beta)| - \alpha^2\, |D_{n-2}(\alpha, \beta)| \tag{3.1}$$

with $|D_0(\alpha, \beta)| = 1$, $|D_1(\alpha, \beta)| = 2\beta$, and

$$2\theta\, |D_n(\alpha, \beta)| = (\beta + \theta)^{n+1} - (\beta - \theta)^{n+1} \tag{3.2}$$

with $\theta = (\beta^2 - \alpha^2)^{1/2}$.

Further, in view of the above relations (3.1) and (3.2), it may be seen easily that

$$2t^{-n}(\beta + \alpha^2 r)\, |D_n(\alpha, \beta)| = 2\beta(1 - (r\alpha)^{2n}) + \alpha^2 r(1 - (r\alpha)^{2n-2}), \tag{3.3}$$

where

$$r = -(1/t) = -\frac{(\beta - (\beta^2 - \alpha^2)^{1/2})}{\alpha^2}.$$

Taking $2\beta = (1 - d(2))$ and $\alpha = -d(2)/2$ in $|D_n(\alpha, \beta)|$ we see from (3.1) that the determinant of the coefficient matrix A of (2.10) satisfies the difference equation

$$|A| = 2\beta\, |D_{n-2}(\alpha, \beta)| - \alpha^2\, |D_{n-3}(\alpha, \beta)|. \tag{3.4}$$

Thus, it follows from (3.3) that

$$2t^{2-n}(\beta + \alpha^2 r)\, |A| = (2\beta + \alpha^2 r)^2 - \alpha^2(1 + 2\beta r)^2\, (\alpha r)^{2(n-3)}. \tag{3.5}$$

Now putting $2\beta=(1-d(2))$ and $\alpha=-d(2)/2$ in (3.5), we can get the elements a_{ij} of A^{-1} from the cofactors of the transpose matrix A. Thus, for $\mathbf{0}<i\leqslant j\leqslant n-2$ (or $i=j=\mathbf{0}$; cf. [1, pp. 35–38])

$$((1-d(2))+r)(1-r^{2n})(a_{ij})=r^{j-i}(1-r^{2i+2})(1-r^{2(n-1-j)}),$$

$$\left((1-d(2))+\frac{r}{2}\right)(1-r^{2n})(a_{in-2})=r^{n-i-2}(1-r^{2(i+1)})\quad\text{for } \mathbf{0}<i\leqslant n-2,$$

$$\left((1-d(2))+\frac{r}{2}\right)(1-r^{2n})(a_{0j})=r^{j}(1-r^{2(n-j-1)})\quad\text{for } \mathbf{0}<j<n-2,$$

$$\left((1-d(2))+\frac{r}{2}\right)^2(1-r^{2n})(a_{0n-2})=r^{n-2}((1-d(2))+r).$$

Clearly A is symmetric; therefore A^{-1} is also symmetric. Now considering a fixed value of x such that $0<x<1$, it may be seen easily that for fixed $\varepsilon>0$ and $\varepsilon<i/n$, $j/n<1-\varepsilon$, the elements a_{ij} of A^{-1} may be approximated asymptotically by $r^{|j-i|}/((1-d(2))+r)$. Further, it can be found in [8] that

$$\sum_i \frac{r^{|j-i|}}{((1-d(2))+r)}=\frac{(1+r)}{(1-r)((1-d(2))+r)},$$

where

$$r=\left(\frac{2}{(d(2))^2}\right)[(1-2d(2))^{1/2}-d(11)].$$

THEOREM 3.1. *For a fixed $\varepsilon>0$ and $\varepsilon<i/n$, $j/n<1-\varepsilon$, the coefficient matrix A of* (2.10) *is invertible and the elements a_{ij} of A^{-1} can be approximated asymptotically by $r^{|j-i|}/((1-d(2))+r)$ and the row max norm of its inverse, that is,*

$$\|A^{-1}\|\leqslant\frac{(1+r)}{(1-r)((1-d(2))+r)}=:K_1,$$

where

$$r=\left(\frac{2}{(d(2))^2}\right)[(1-2d(2))^{1/2}-d(11)]. \tag{3.6}$$

Remark 3.1. In studies concerning discrete splines smaller values of h have special significance for the simple reason that discrete splines reduce to continuous splines as $h\to 0$. It is interesting to note that the estimate

(3.6) is sharper than that obtained in terms of the infimum of the excess of the absolute value of the leading diagonal element over the sum of the absolute values of other elements in each row.

For the latter of these it gives

$$\|A^{-1}\| \leqslant 3, \tag{3.7}$$

and a simple computation shows that

$$3 > K_1.$$

4. Error Bounds

In this section, we shall obtain the error bounds for the deficient discrete cubic spline interpolant, i.e., $e = f - s$ over the discrete interval $[0, 1]_h$ which is defined by $[0, 1]_h = [0, 1] \cap R_h$, where $R_h = \{x_0 + jh\colon j \text{ is an integer}\}$.

In order to show the convergence of the discrete spline, we observe that we do not require any smoothness condition on the function, as is required in the continuous spline case.

We shall need the following Lemma due to Lyche [5].

Lemma 4.1. *Let $\{a_j\}_{j=1}^m$ and $\{b_j\}_{j=1}^n$ be given sequences of nonnegative real numbers such that $\sum_{j=1}^m a_j = \sum_{j=1}^n b_j$; then for any real valued function f, defined on a discrete interval $[0, 1]_h$, we have*

$$\left| \sum_{j=1}^{m} a_j [x_{j0}, x_{j1}, ..., x_{jk}] f - \sum_{j=1}^{n} b_j [y_{j0}, y_{j1}, ..., y_{jk}] f \right|$$

$$\leqslant W(D_h^{\{k\}} f, |1 - kh|) \sum \frac{a_j}{k!},$$

where $x_{jk}, y_{jk} \in [0, 1]_h$ for relevant values of j, k.

Replacing m_i by $D_h^{\{1\}} e(x_i)$ in Eq. (2.8) we have

$$e(x) = n^2 [y_i D_h^{\{1\}} e(x_i) + y_i^1 D_h^{\{1\}} e(x_{i-1}) - T_i(f)], \tag{4.1}$$

where $T_i(f) = [y_i D_h^{\{1\}} f(x_i) + y_i^1 D_h^{\{1\}} f(x_{i-1}) + (9n/2)(f(\beta_i)\,\mathbf{Y}_i - f(\alpha_i)\,\mathbf{Y}_i^1) - f(x)/n^2]$.

In order to find the bound of $f(x)-s(x)$, first we estimate $|T_i(f)|$, writing it in the form

$$|T_i(f)|=\left|\sum_{j=1}^{3} a_j[x_{j0}, x_{j1}]f-\sum_{j=1}^{4} b_j[y_{j0}, y_{j1}]f\right|,$$

where $a_1=Q_{i-1}(x)$, $a_2=\frac{3}{2}((d^*(7)(a_1+a_3))+n^{-1}(x_i-x)(x-\beta_i))$, $a_3=Q_i(x)$, and $b_1=b_4=d(2)(a_1+a_3)$, $b_2=3a_1$, $b_3=(x-\beta_i)/n^2$, $x_{10}=y_{10}=x_{i-1}-h$, $x_{11}=y_{11}=x_{i-1}+h$, $x_{20}=y_{20}=\alpha_i=x_{i-1}+1/3n$, $x_{21}=y_{21}=\beta_i=x_{i-1}+2/3n=y_{40}$, $y_{41}=x$, $x_{30}=y_{30}=x_i-h$, and $x_{31}=y_{31}=x_i+h$.

It can be seen easily that $\Sigma a_j=\Sigma b_j$. Therefore, applying Lemma 4.1 for $n=3$, $m=4$, and $k=1$, we get

$$|T_i(f)|\leqslant K_2\, W(D_h^{\{1\}}f, 1/n), \tag{4.2}$$

where $K_2=(n^{-3}/1125)[8\{3d(0)+d^*(37)\}-75]$.

We now proceed to obtain an upper bound for $D_h^{\{1\}}e(x_i)$. Replacing m_i by $D_h^{\{1\}}e(x_i)$ in Eq. (2.10), we have

$$A(D_h^{\{1\}}e(x_i))=A(D_h^{\{1\}}f(x_i))-(F_i)=:(H_i), \qquad i=1, 2, ..., n-1. \tag{4.3}$$

Again, observe that (H_i) has the form

$$\left|\sum_{j=1}^{3} a_j[x_{j0}, x_{j1}]f-\sum_{j=1}^{3} b_j[y_{j0}, y_{j1}]f\right|,$$

where $a_1=d(11)$, $a_2=a_3=(3/4)\,d^*(7)$, $b_2=3/2$, $b_1=b_3=d(2)/2$, $x_{10}=x_i-h$, $x_{11}=x_i+h$, $y_{10}=x_{i-1}-h$, $y_{11}=x_{i-1}+h$, $x_{20}=\alpha_i=x_{i-1}+1/3n$, $y_{20}=\beta_i=x_{i-1}+2/3n=x_{21}$, $y_{21}=\alpha_{i+1}=x_i+1/3n=x_{30}$, $y_{30}=x_{i+1}-h$, $y_{31}=x_{i+1}+h$, $x_{31}=\beta_{i+1}=x_i+2/3n$.

Clearly $\Sigma a_j=\Sigma b_j$; thus by using Lemma 4.1 again, we get

$$|(H_i)|\leqslant K_3\, W(D_h^{\{1\}}f, 1/n), \tag{4.4}$$

where

$$K_3=\frac{3(72h^2+43/n^2)}{2(18h^2+13/n^2)}.$$

Thus, using Eqs. (4.3)–(4.4) and (3.6), we get

$$\|(D_h^{\{1\}}e(x_i))\|\leqslant K_4\, W(D_h^{\{1\}}f, 1/n), \tag{4.5}$$

where $K_4 = K_1 K_3$ and, finally, using the bounds of (4.2) and (4.5) in (4.1), we have

$$\|e(x)\| \leqslant n^{-1} K(h)\ W(D_h^{\{1\}} f, 1/n),$$

where $K(h) = ((1 - 2d(2))\ M + K_2)$, with $M = (4/1125)\ K_4$. Thus we have proved the following.

THEOREM 4.1. *Suppose* $s(x, h)$ *is the* 1*-periodic deficient discrete cubic spline interpolant of Theorem* 2.1. *Then over the discrete interval* $[0, 1]_h$

$$\|e(x)\| \leqslant n^{-1} K(h)\ W(D_h^{\{1\}} f, 1/n), \tag{4.6}$$

where $K(h)$ *is some function of h defined earlier and* $W(f, 1/n)$ *is the discrete modulus of the continuity of f.*

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